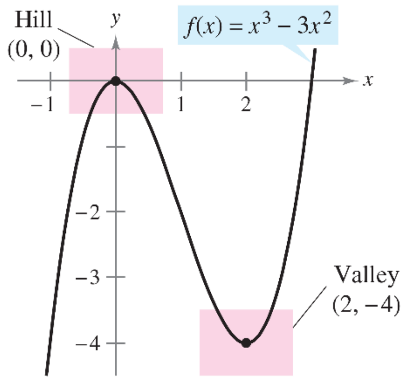
Extrema on an Interval 区间上的极值(p164)

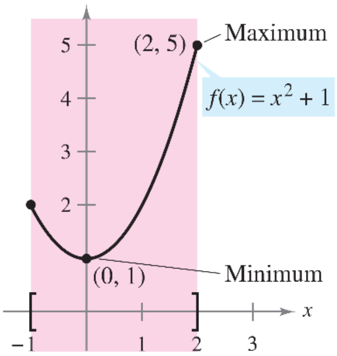
区间上的**极小值**：$c \in (a,b), f(c) \le f(x), f(c)$ is the relative minimum.
区间上的**极大值**：$c \in (a,b), f(c) \ge f(x), f(c)$ is the relative maximum.
极大值和极小值也称之为区间极大极小值(local minimum/local maximum)

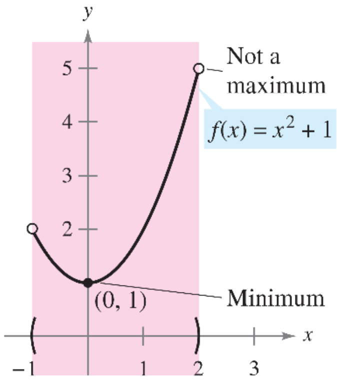


f has a relative maximum at (0, 0) and a relative minimum at (2, −4).

中文里极值是指的某个区间的极大或极小值(relative extrema)，**最值**是指整个函数的最大和最小值(**Absolute Extrema**)。

最大值
- 极值中最大的
- 函数的端点
- 不连续的点
- 不可导的点





极值存在定理 The Extreme Value Theorem（P164） 不重要

EXAMPLE 1 The Value of the Derivative at Relative Extrema

Find the value of the derivative at each relative extremum shown in Figure 3.3.

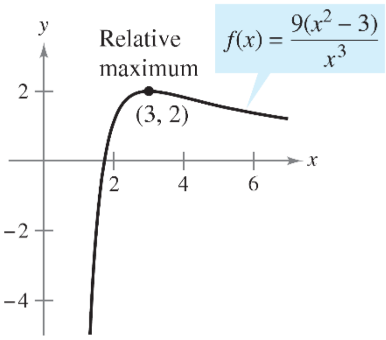
Solution

a. The derivative of $f(x) = \dfrac{9(x^2 - 3)}{x^3}$ is

$$f'(x) = \frac{x^3(18x) - (9)(x^2 - 3)(3x^2)}{(x^3)^2} \qquad \text{Differentiate using Quotient Rule.}$$

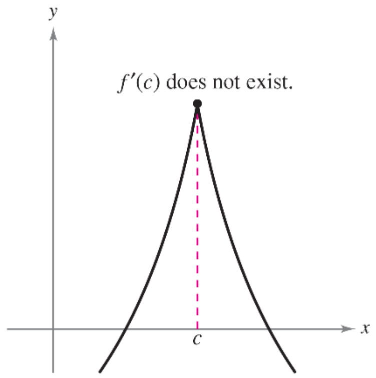
$$= \frac{9(9 - x^2)}{x^4}. \qquad \text{Simplify.}$$

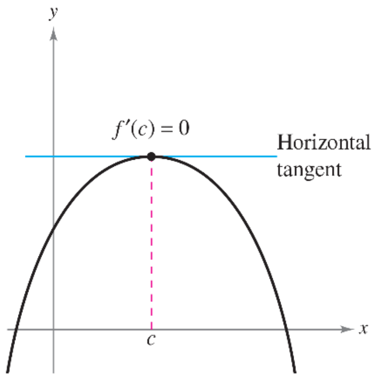
At the point $(3, 2)$, the value of the derivative is $f'(3) = 0$



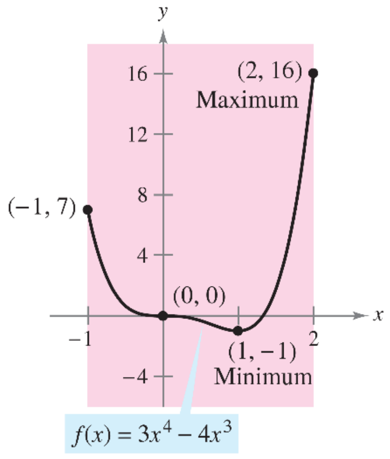
Critical number $c \begin{cases} f'(c) = 0 \\ not\ differentiable\ at\ c \end{cases}$

The Relative Extrema occur only at Critical Numbers.(p166)





Example 2: Find Extrema of $f(x)=3x^4-4x^3$ on the interval $[1,2]$



Solution

$$f(x)=3x^4-4x^3$$

$$f'(x)=12x^3-12x^2=0\Rightarrow x=0,1$$

Left endpoint: $f(-1)=7$

Critical number: $f(0)=0$

Critical number: $f(1)=-1$ Minimum

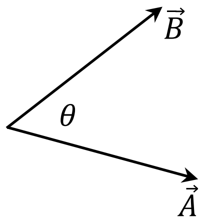
Right endpoint: $f(2)=16$ Maximum

复习：向量$\vec{A}=a_x\vec{\imath}+a_y\vec{\jmath}$，$\vec{B}=b_x\vec{\imath}+b_y\vec{\jmath}$

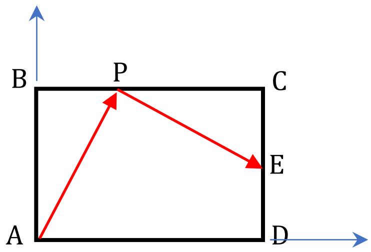
数量积的定义：$\vec{A}\cdot\vec{B}=|\vec{A}|\cdot|\vec{B}|\cos\theta$

Coordinate Form (2D)

$$\begin{cases} magnitude: & \vec{A}\cdot\vec{B}=a_xb_x+a_yb_y \\ direction: & \cos\theta=\dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}|\cdot|\vec{B}|}=\dfrac{a_xb_x+a_yb_y}{|\vec{A}|\cdot|\vec{B}|} \end{cases}$$



Example 3: 矩形 ABCD 如图所示，$\overline{AB}=2,\overline{BC}=3$，E 为 CD 中点，P 为 BC 上一动点。



1. 建立坐标系如图所示，设$\overline{BP}=t$，写出A,B,C,D,E,P 点的坐标。

$A\ (0,0)$ $B\ (0,2)$ $C\ (3,2)$ $D\ (3,0)$ $E\ (3,1)$

$P\ (t,2), t\epsilon[0,3]$

2. 写出向量$\overrightarrow{AP}$与$\overrightarrow{PE}$的表达式

$\overrightarrow{AP}=t\vec{\imath}+2\vec{\jmath}$ $\overrightarrow{PE}=(3-t)\vec{\imath}-\vec{\jmath}$

3. 求$\overrightarrow{AP}\cdot\overrightarrow{PE}$的最大值

$$\overrightarrow{AP}\cdot\overrightarrow{PE}=t(3-t)-2=-t^2+3t-2 \qquad t\epsilon[0,3]$$

$$\frac{d}{dt}(\overrightarrow{AP}\cdot\overrightarrow{PE})=\frac{d}{dt}(-t^2+3t-2)=-2t+3=0\Rightarrow t=3/2\epsilon[0,3]$$

Left endpoint: $f(0)=-2$

Critical number: $f(3/2)=1/4$ Maximum

Right endpoint: $f(3)=-5$

Exercise: Locate the absolute extrema of the functions on the closed interval.

21. $f(x)=x^3-\frac{3}{2}x^2$ $[-1,2]$

习题答案

$$f'(x)=3x^2-3x=0\ \Rightarrow\ x=0,1$$

Left endpoint: $f(-1)=-5/2$ Minimum

Critical number: $f(0)=0$

Critical number: $f(1)=-1/2$

Right endpoint: $f(2)=2$ Maximum

22. $f(x)=x^3-12x$ $[0,4]$

$f^{'}(x)=3x^2-12=0 \Rightarrow x=-2,2$

Left endpoint: $f(0)=1/2$

Critical number: $f(-2)$ not in the interval

Critical number: $f(2)=-16$ Minimum

Right endpoint: $f(4)=16$ Maximum

25. $g(t)=\frac{t^2}{t^2+3}$ $[-1,1]$

$g(t)=\frac{t^2}{t^2+3}=\frac{t^2+3-3}{t^2+3}=1-\frac{3}{t^2+3}=1-3(t^2+3)^{-1}$

$g^{'}(t)=-3(-1)(t^2+3)^{-2}(2t)=6t/(t^2+3)^2=0 \Rightarrow t=0$

Left endpoint: $g(-1)=1/4$ Maximum

Critical number: $g(0)=0$ Minimum

Right endpoint: $g(1)=1/4$ Maximum

26. $f(x)=\frac{2x}{x^2+1}$ $[-2,2]$

$f^{'}(x)=\frac{2(x^2+1)-2x(2x)}{x^2+1}=\frac{2-2x^2}{x^2+1}=0 \Rightarrow x=-1,1$

Left endpoint: $f(-2)=-4/5$ Minimum

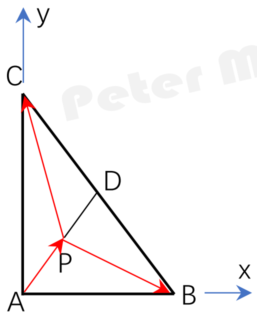
Critical number: $f(-1)=-1$

Critical number: $f(1)=1$

Right endpoint: $f(2)=4/5$ Maximum

Exercise: 在直角三角形 ABC 中，∠A=90°，AB=3,AC=4.D 是 BC 中点，P 是 AD 上一点。求$\overrightarrow{AP}\cdot(\overrightarrow{PB}+\overrightarrow{PC})$的最大值。

Solution: 设 置 坐 标 系 如 图 ， 所 以 三 角 形 三 个 顶 点 的 坐 标 为 A(0,0),B(3,0),C(0,4)。D 为 BC 中点，所以 D(3/2,2)。



设$\overrightarrow{AP}=t\overrightarrow{AD}=t\left(\frac{3}{2}\vec{\imath}+2\vec{\jmath}\right)=\frac{3t}{2}\vec{\imath}+2t\vec{\jmath}$ $t\in[0,1]$

$\overrightarrow{PB}=\left(3-\frac{3t}{2}\right)\vec{\imath}-2t\vec{\jmath}$, $\overrightarrow{PC}=\left(\frac{-3t}{2}\right)\vec{\imath}+(4-2t)\vec{\jmath}$

$\overrightarrow{PB}+\overrightarrow{PC}=\left(3-\frac{3t}{2}\right)\vec{\imath}-2t\vec{\jmath}+\left(\frac{-3t}{2}\right)\vec{\imath}+(4-2t)\vec{\jmath}=$

$(3-3t)\vec{\imath}+(4-4t)\vec{\jmath}$

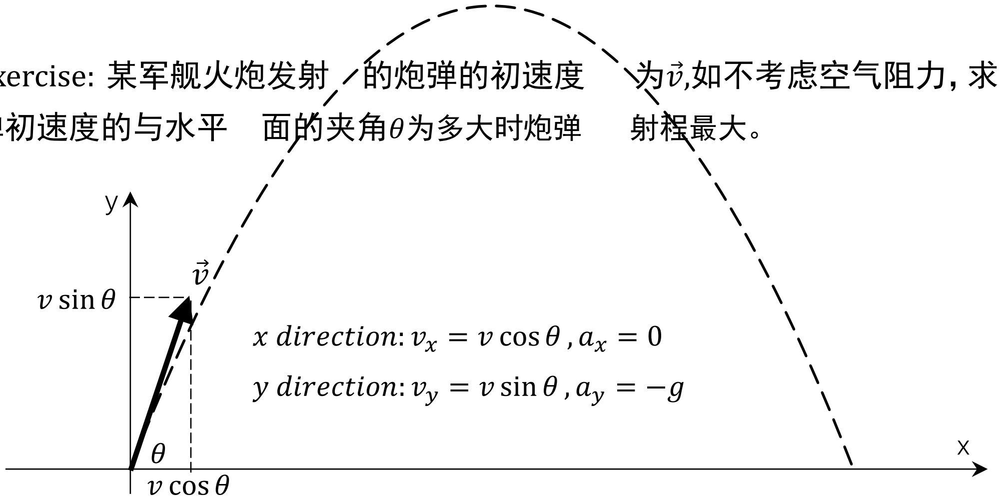
$\overrightarrow{AP}\cdot(\overrightarrow{PB}+\overrightarrow{PC})=\left(\frac{3t}{2}\vec{\imath}+2t\vec{\jmath}\right)\cdot[(3-3t)\vec{\imath}+(4-4t)\vec{\jmath}]$

$=\left(\frac{3t}{2}\right)(3-3t)+2t(4-4t)=\frac{25}{2}t-\frac{25}{2}t^2=f(t)$

$f'(t)=\frac{25}{2}-\frac{25}{2}(2t)=0\ \Rightarrow \mathrm{t}=\frac{1}{2}$

Left endpoint:	$f(0)=0$	
Critical number:	$f(1/2)=25/8$	Maximum
Right endpoint:	$f(1)=0$	

Exercise: 某军舰火炮发射的炮弹的初速度为$\vec{v}$,如不考虑空气阻力，求炮弹初速度的与水平面的夹角θ为多大时炮弹射程最大。



Solution

y 方向: $v_{y0}=v\sin\theta,\ v_{yt}=-v_{y0}$时炮弹落地

$$-v_{y0}=v_{y0}-gt\ \Rightarrow\ 2v_{y0}=gt=2v\sin\theta\Rightarrow t=(2v/g)sin\,\theta$$

x 方向：$x_t=v_{x0}t=v\cos\theta\left(\frac{2v}{g}\right)\sin\theta=\left(\frac{v^2}{g}\right)(2\sin\theta\cos\theta)$

$$=(v^2/g)sin\,2\theta\quad \theta\epsilon[0,\pi/2]$$

$x'_t=(v^2/g)2cos\,2\theta=0\ \Rightarrow\ \theta=\ \pi/4$

Left endpoint:	$x(0)=0$	
Critical number:	$x(\pi/4)=v^2/g$	Maximum
Right endpoint:	$x(\pi)=0$	

总结—求最值/极值问题

1. 对所求函数或方程，找到一个变量或参数进行表达
2. 大小固定角度变化的情况，经常用变化的角度θ为参数
   角度固定(某直线上的动点)，经常用相对长度 t 作为参数
3. 明确参数的取值范围进行求解

Exercise:尝试找出相关的参数表达下面各变量及其变化范围，不必求解

1. 矩形 ABCD，AB=3, BC=4. E 是 BC 中点，P 是矩形内一点且 AP=1。求线段 PD+PE 的长度的最大值和最小值。

2. 平行四边形 ABCD，$\angle A = \frac{\pi}{3}, AB = 3, AD = 2$, P 为 DC 上一点，求$\overrightarrow{AP} \cdot \overrightarrow{PB}$的最大值和最小值。

3. 某海边炮台高于海平面 h，炮弹初速度大小为 v，求最大射程。

Exercise: Locate the absolute extrema of the functions on the closed interval.

23. $f(x) = x^3 - \frac{3}{2}x^2$ $[-1,2]$

24. $f(x) = x^3 - 12x$ $[0,4]$

27. $g(t) = \frac{t^2}{t^2+3}$ $[-1,1]$

28. $f(x) = \frac{2x}{x^2+1}$ $[-2,2]$

Exercise: 在直角三角形 ABC 中，∠A=90°，AB=3,AC=4.D 是 BC 中点，P 是 AD 上一点。

复习：向量$\vec{A}=a_x\vec{\imath}+a_y\vec{\jmath}$，$\vec{B}=b_x\vec{\imath}+b_y\vec{\jmath}$

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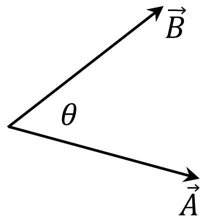

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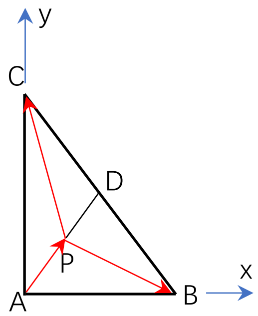
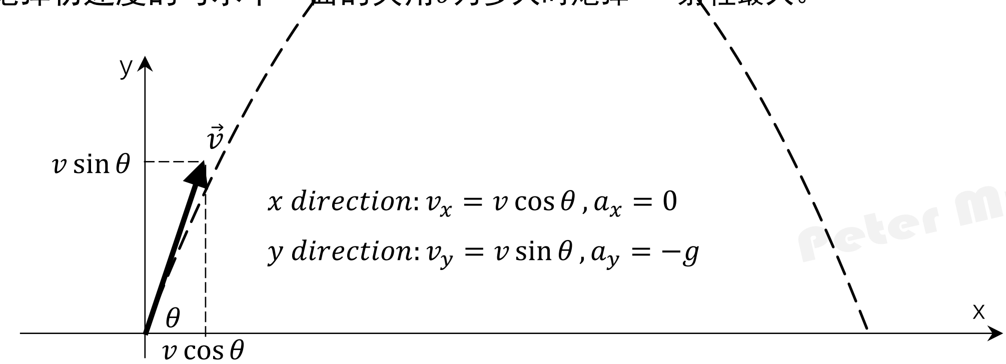
(a) 设置坐标系如图，写出 A,B,C,D 的坐标。



(b) 设$\overrightarrow{AP}=t\overrightarrow{AD}$, 写出$\overrightarrow{AD},\overrightarrow{AP},\overrightarrow{PB}$和$\overrightarrow{PC}$的表达式

(c) 求$\overrightarrow{AP}\cdot(\overrightarrow{PB}+\overrightarrow{PC})$的最大值。

Exercise: 某军舰火炮发射的炮弹的初速度为$\vec{v}$, 如不考虑空气阻力，求炮弹初速度的与水平面的夹角θ为多大时炮弹射程最大。



总结—求最值/极值问题
1. 对所求函数或方程，找到一个变量或参数进行表达
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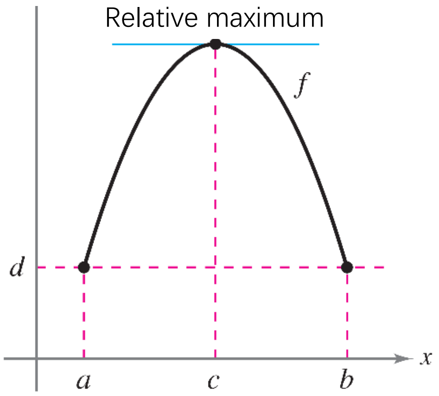
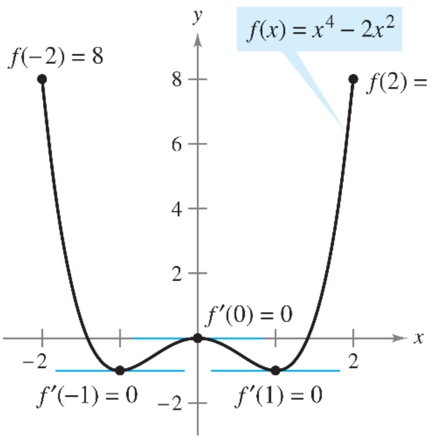
3. 某海边炮台高于海平面 h, 炮弹初速度大小为 v, 求最大射程。

Rolle's Theorem 罗尔定理(p172)



$f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$, then there is at least one number c in (a,b) such that $f'(c)=0$.



Example $f(x)=x^4-2x^2$. Find all values of c in $(-2,2)$ such that $f'(c)=0$.

Solution $f'(x)=4x^3-4x \Rightarrow$

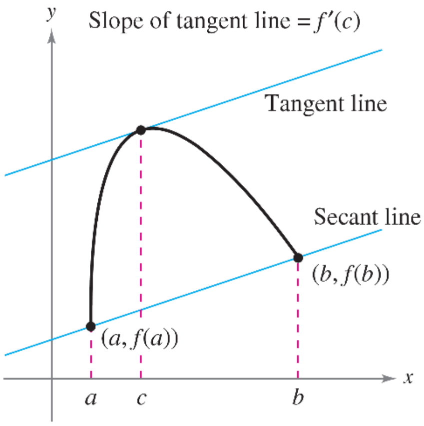
$$\left.\begin{array}{r} \because f(x)\text{is continuous in } [-2,2] \\ \because f(x)\text{is differentiable in } (-2,2) \\ f(-2)=f(2)=8 \end{array}\right\} \xRightarrow{Rolle's\ theorem}$$

$\Rightarrow \exists c\epsilon(-2,2), f'(c)=0$

$f'(x)=4x^3-4x=0 \Rightarrow x=0,-1,1$

罗尔定理是否适用：三个条件要找全!

The Mean Value Theorem (p174)



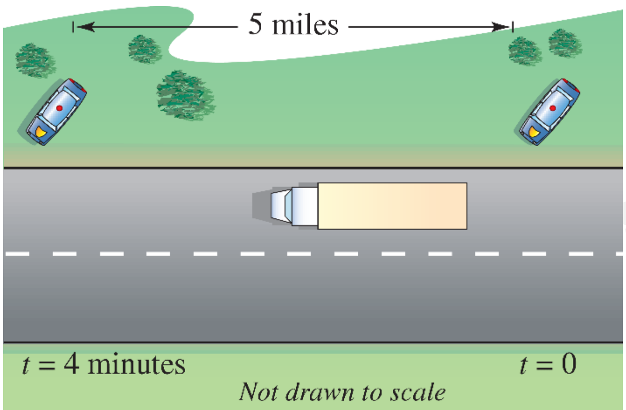
$f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there is at least a number c in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Useful alternative form

$$f(b)=f(a)+(b-a)f'(c)$$

Example Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in the figure. As a truck passes the first car, its speed is clocked at 55 mph. 4 minutes later, when the truck passes the second patrol car, its speed is clocked at 50 mph. Prove that the truck must have exceed the speed limit (of 55 mph) at some time during the 4 minutes.



<u>*Solution*</u> Let $t=0$ be the time when the truck passes the first car. When it passes the second car, $t=4/60=1/15$ hour.

Let $s(t)$ be the distance traveled by the truck, so

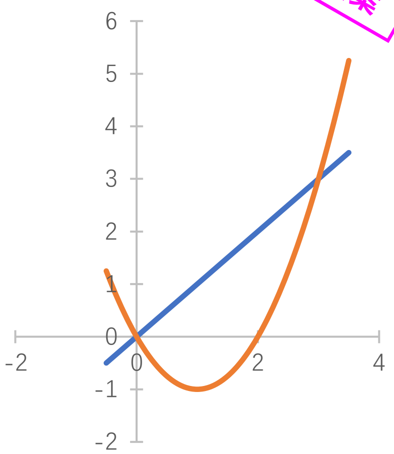
$s(0) = 0$ and $s(1/15) = 5$.

$$V_{avg} = \frac{s(1/15)-s(0)}{(1/15)-0} = \frac{5}{1/15} = 75 \text{ mph}$$

Assuming that the position function is differentiable, according to the mean value theorem, the truck must have been traveling at a rate of 75 mph during the 4 minutes.

Exercise: $y_1 = x, y_2 = (x-1)^2 - 1 = x^2 - 2x$

1. y_1 and y_2 intercept at A and B, find the coordinate of A and B.
2. Point C is on y_2 and between A and B. Find the coordinate of C to satisfy that △ABC has the maximum space.



Solution1: 初中数学方法

1. 求解得$A(0,0), B(3,3)$

2. C 在y_2上，当过 C 点，和直线y_1平行，且

   和y_2相切时，$S_{\triangle ABC}$取最大值。

   y_1的斜率是 1，所以和y_1平行的直线方程为$y_3 = x + t, t$为参数。

   求y_2和y_3的交点

   $$x^2 - 2x = x + t \Rightarrow x^2 - 3x - \mathrm{t} = 0$$

   当y_2与y_3相切时，只有 1 个交点，此时上述二次方程$\Delta = 0$

   $$\Delta = (-3)^2 - 4(-t) = 0 \Rightarrow \mathrm{t} = -9/4$$

   $$x = \frac{-b}{2a} = \frac{3}{2} \qquad y_3 = x + t = \frac{3}{2} - \frac{9}{4} = \frac{-3}{4}$$

   $C\left(\frac{3}{2}, \frac{-3}{4}\right)$，直线 BC 交 x 轴于点$D\left(\frac{9}{5}, 0\right)$

   $$S_{\triangle ABC} = S_{\triangle ABD} + S_{\triangle ACD} = \frac{1}{2} \cdot \frac{9}{5} \cdot 3 + \frac{1}{2} \cdot \frac{9}{5} \cdot \frac{3}{4} = \frac{27}{8}$$

Solution2: The Mean Value Theorem

$$\left.\begin{array}{r} y_2'(x) = 2x - 2 \\ y_2'(c) = \frac{y_2'(b) - y_2'(a)}{b-a} = \frac{3-3}{3-3} = 1 \end{array}\right\} \Rightarrow y_2'(x) = 2x - 2 = 1 \Rightarrow x = \frac{3}{2}$$

利用$y_2 = x^2 - 2x, x = \frac{3}{2}$可得切点坐标$C\left(\frac{3}{2}, \frac{-3}{4}\right)$

Exercise: Determine whether Rolle's theorem can be applied to $f(x)$ on the interval and , if so, find all values of c in the interval such that $f'(c) = 0$.

25. $f(x) = |x| - 1 \qquad [-1,1]$

$f(x)$ is not differentiable at $(0,0)$ so Rolle's theorem does not apply.

26. $f(x) = x - x^{\frac{1}{3}} \qquad [0,1]$

$f(x) = x - x^{\frac{1}{3}}$ In $[0,1]$ continuous

$f'(x) = 1 - \frac{1}{3}\left(x^{\frac{-2}{3}}\right) = 1 - \frac{1}{3\sqrt[3]{x^2}}$ In $(0,1)$ differentiable

$f(0) = f(1) = 0$ $f(b) = f(a)$

Rolle's theorem can be applied.

$f'(c) = 1 - \frac{1}{3\sqrt[3]{c^2}} = 0 \Rightarrow c = \frac{\pm 1}{3\sqrt{3}} = \pm\frac{\sqrt{3}}{9}$

27. $f(x) = x - \tan \pi x \quad [\frac{-1}{4}, \frac{1}{4}]$

$f(x) = x - \tan \pi x$ In $[\frac{-1}{4}, \frac{1}{4}]$ continuous

$f'(x) = 1 - \pi \sec^2 \pi x$ In $\left(\frac{-1}{4}, \frac{1}{4}\right)$ differentiable

$f\left(\frac{-1}{4}\right) \neq f\left(\frac{1}{4}\right)$ $f(b) \neq f(a)$

Rolle's theorem does not apply.

28. $f(x) = \frac{x}{2} - \sin\frac{\pi x}{6} \quad [-1,0]$

$f(x) = \frac{x}{2} - \sin\frac{\pi x}{6}$ In $[-1,0]$ continuous

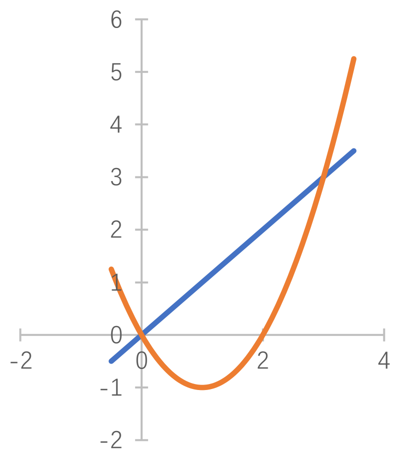
$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\frac{\pi x}{6}$ In $(-1,0)$ differentiable

$f(-1) \neq f(0)$ $f(b) \neq f(a)$

Rolle's theorem does not apply.

Exercise: $y_1 = x, y_2 = (x-1)^2 - 1 = x^2 - 2x$

1. y_1 and y_2 intercept at A and B, find the coordinate of A and B.
2. Point C is on y_2 and between A and B. Find the coordinate of C to satisfy that △ABC has the maximum space.



Exercise: Determine whether Rolle's theorem can be applied to $f(x)$ on the interval and , if so, find all values of c in the interval such that $f^{'}(c) = 0$.

25. $f(x) = |x| - 1$ $[-1,1]$

26. $f(x) = x - x^{\frac{1}{3}}$ $[0,1]$

27. $f(x) = x - \tan \pi x$ $[\frac{-1}{4}, \frac{1}{4}]$

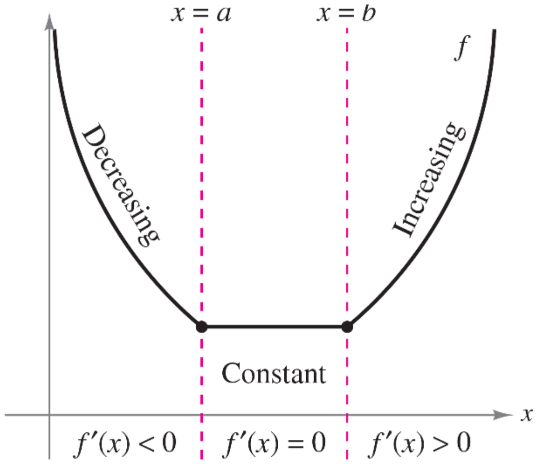
28. $f(x) = \frac{x}{2} - \sin \frac{\pi x}{6}$ $[-1,0]$

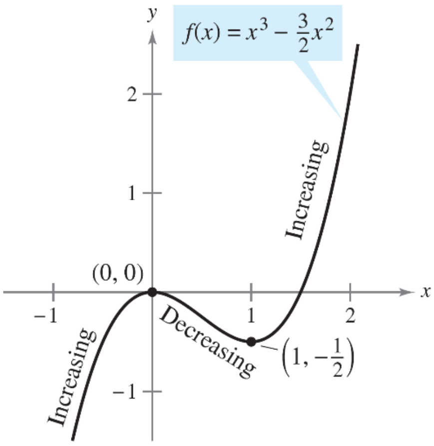
Increasing and decreasing function 增函数和减函数(p179)

$f'(x) < 0$ Decreasing

$f'(x) = 0$ Constant

$f'(x) > 0$ Increasing

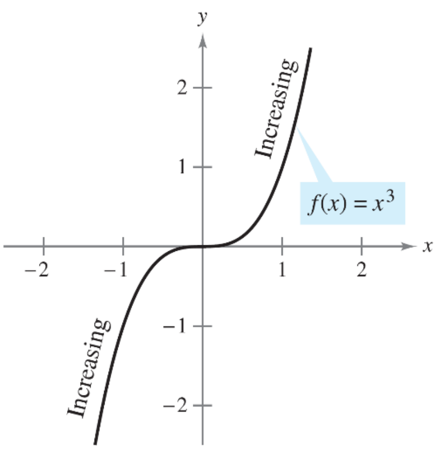




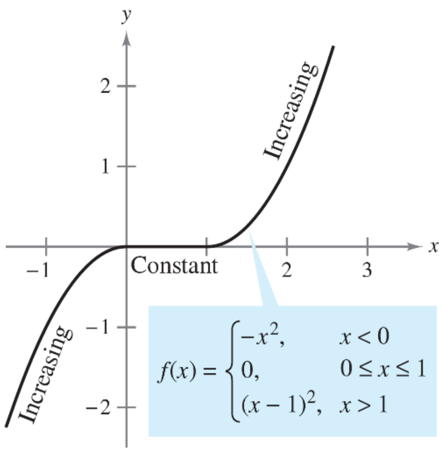
Example: Find out each monotonic interval of $f(x) = x^3 - \frac{3}{2}x^2$

$f(x) = x^3 - \frac{3}{2}x^2$ Domain: $x \epsilon R$				Step 1: Check Domain
$f'(x) = 3x^2 - 3x = 0$				Step 2: $f'(x) = 0$
$x = 0,1$				Step 3: Critical no
Interval	$(-\infty, 0)$	$(0,1)$	$(1, \infty)$	Step 4: Test $f'(x)$ in each interval
Test Value	$x = -1$	$x = 1/2$	$x = 2$	
$f'(x)$	$6 > 0$	$-3/4 < 0$	$6 > 0$	
Conclusion	Inc	Dec	Inc	

Notice two types of monotonic

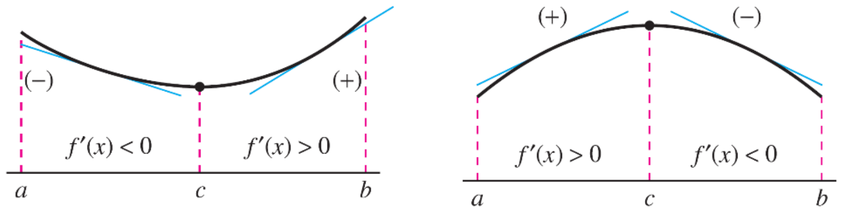


(a) Strictly monotonic function

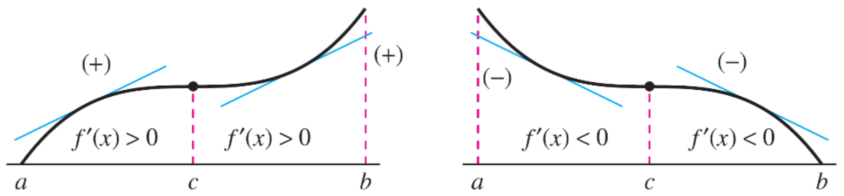


(b) Not strictly monotonic

If $f'(x)$ is positive on both side of c, it is not a extrema. Same for both negative.



Relative minimum Relative maximum



Neither relative minimum nor relative maximum

Example: Find the relative extrema of $f(x) = (x^2-4)^{2/3}$.

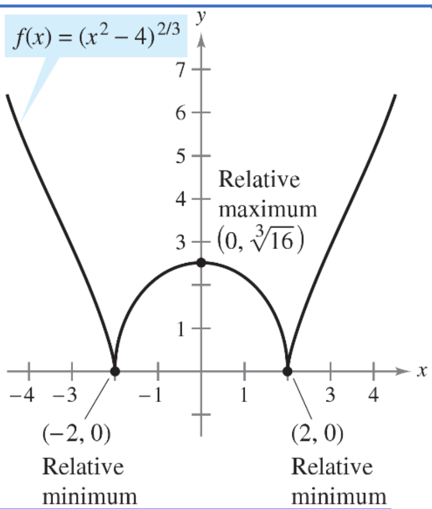
Solution

$f(x) = (x^2-4)^{2/3}$ Domain: $x \in R$

$f'(x) = \frac{2}{3}(x^2-4)^{-\frac{1}{3}}(2x)$

$= \frac{4x}{3(x^2-4)^{1/3}} = 0 \Rightarrow x = 0$

$\Rightarrow f'(x)$ does not exist when $x \pm 2$



Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test value	$x = -3$	$x = -1$	$x = 1$	$x = 3$
Sign of $f'(x)$	$f'(-3) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

Exercise: Find the relative extrema of

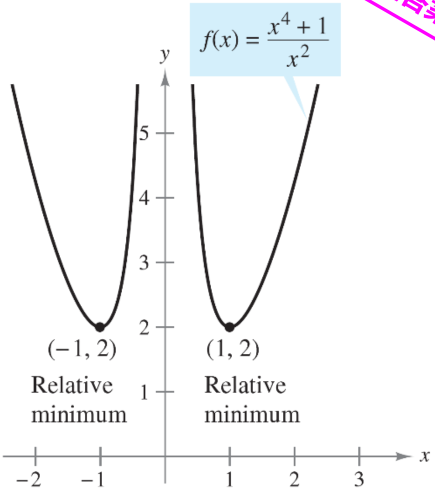
$$f(x) = \frac{x^4+1}{x^2}.$$

Solution

$$f(x) = \frac{x^4+1}{x^2} \qquad \text{Domain: } x \neq 0$$

$$f'(x) = \frac{2(x^4+1)}{x^3} = \frac{2(x^2+1)(x-1)(x+1)}{x^3}$$

$$f'(x) = 0 \Rightarrow x = 0(NG), x = \pm 1$$



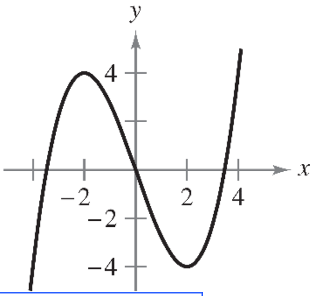
Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test value	$x = -2$	$x = -1/2$	$x = 1/2$	$x = 2$
Sign of $f'(x)$	$f'(-2) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(2) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

Exercise: Identify the open intervals on which the function is increasing or decreasing.

5. $y = \frac{x^3}{4} - 3x$

$$y = \frac{x^3}{4} - 3x \quad \text{Domain } x \in R$$

$$y' = \frac{3x^2}{4} - 3 = 0 \Rightarrow \text{x} = \pm 2$$

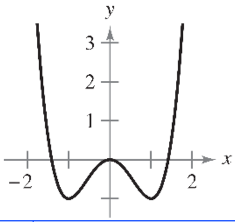


Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test value	$x = -4$	$x = 0$	$x = 4$
Sign of $f'(x)$	$f'(-4) > 0$	$f'(0) < 0$	$f'(4) > 0$
Conclusion	Increasing	Decreasing	Increasing

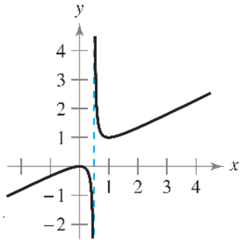
6. $f(x) = x^4 - 2x^2$

$$f(x) = x^4 - 2x^2 \quad \text{Domain } x \in R$$

$$f'(x) = 4x^3 - 4x = 0 \Rightarrow x = 0, x = \pm 1$$



Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test value	$x = -2$	$x = -1/2$	$x = 1/2$	$x = 2$
Sign of $f'(x)$	$f'(-2) < 0$	$f'\left(\frac{-1}{2}\right) > 0$	$f'\left(\frac{1}{2}\right) < 0$	$f'(2) > 0$



Conclusion	Decreasing	Increasing	Decreasing	Increasing

7. $f(x) = \frac{1}{(x+1)^2}$

$f(x) = \frac{1}{(x+1)^2}$ Domain $x \neq -1$

$f'(x) = -2(x+1) = 0 \Rightarrow x = -1$ (out of domain)

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Test value	$x = -2$	$x = 0$
Sign of $f'(x)$	$f'(-4) > 0$	$f'(0) < 0$
Conclusion	Increasing	Decreasing

8. $y = \frac{x^2}{2x-1}$

$y = \frac{x^2}{2x-1}$ Domain $x \neq 1/2$

$y' = \frac{2x^2-2x}{(2x-1)^2} = 0 \Rightarrow x = 0,1$

Interval	$-\infty < x < 0$	$0 < x < \frac{1}{2}$	$\frac{1}{2} < x < 1$	$1 < x < \infty$
Test value	$x = -1$	$x = -1/4$	$x = 3/4$	$x = 2$
Sign of $f'(x)$	$f'(-2) > 0$	$f'\left(\frac{-1}{2}\right) < 0$	$f'\left(\frac{1}{2}\right) > 0$	$f'(2) < 0$
Conclusion	Increasing	Decreasing	Increasing	Decreasing

Exercise: Find the relative extrema of

$f(x) = \frac{x^4+1}{x^2}$.

Solution

Interval
Test value
Sign of $f'(x)$
Conclusion

Exercise: Identify the open intervals on which the function is increasing or decreasing.

5. $y = \frac{x^3}{4} - 3x$

Interval

Test value

Sign of $f'(x)$

Conclusion

6. $f(x) = x^4 - 2x^2$

Interval

Test value

Sign of $f'(x)$

Conclusion

7. $f(x) = \frac{1}{(x+1)^2}$

Interval

Test value

Sign of $f'(x)$

Conclusion

8. $y = \frac{x^2}{2x-1}$

Interval

Test value

Sign of $f'(x)$

Conclusion

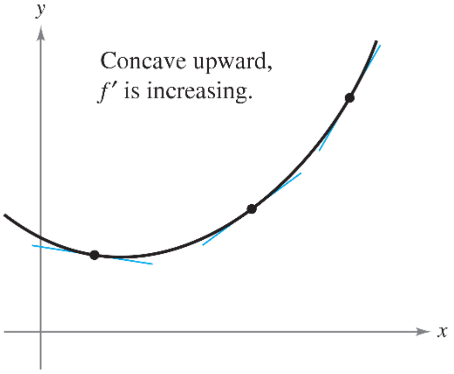
Concavity 函数的凸凹性(p190)

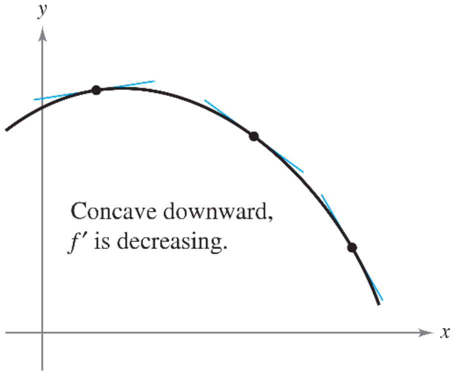
Concave upward 凹函数

$f'(x)$ is increasing

Concave downward 凸函数

$f'(x)$ is increasing

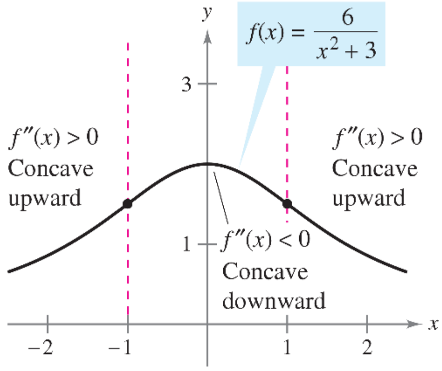




Example1: Determine the open intervals on which the graph of

$$f(x) = \frac{6}{x^2+3}$$

is concave upward or downward.



Solution

$f(x) = \frac{6}{x^2+3}$ Domain: $x\epsilon R$ Step1. Domain

$f'(x) = (-6)(x^2+3)^{-2}(2x) = \frac{-12x}{(x^2+3)^2}$ Domain: $x\epsilon R$ Step2. $f'(x)$

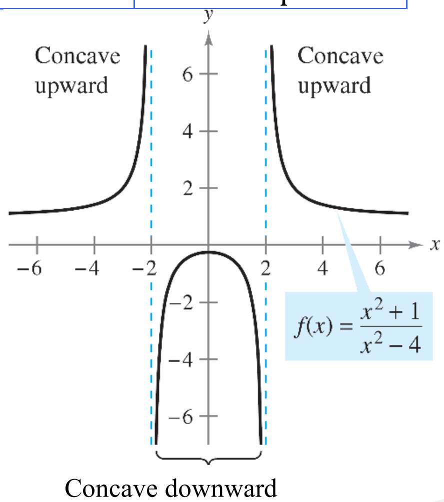
$f''(x) = \frac{36(x^2-1)}{(x^2+3)^3} = 0, x = \pm 1$ Domain: $x\epsilon R$ Step3. $f''(x)$

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Example2: Determine the open intervals on which the graph of

$$f(x) = \frac{x^2+1}{x^2-4}$$

is concave upward or downward.



Solution

$f(x) = \frac{x^2+1}{x^2-4}$ Domain: $x \neq \pm 2$ Step1. Domain

$f'(x) = \frac{-10x}{(x^2-4)^2}$ Domain: $x \neq \pm 2$ Step2. $f'(x)$

$f''(x) = \frac{10(3x^2+4)}{(x^2-4)^3} = 0$ No point at which $f''(x) = 0$ Step3. $f''(x)$

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test value	$x = -3$	$x = 0$	$x = 3$

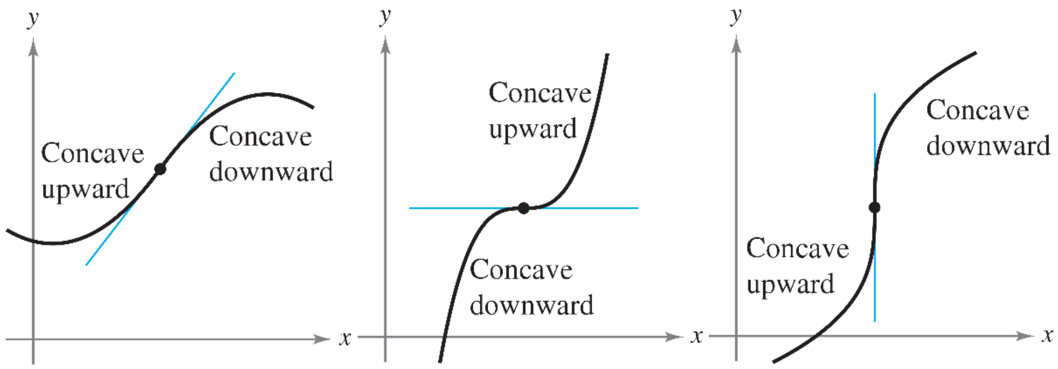
Sign of $f'(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Point of Inflection 拐点(p192)

The graph in Example 1 $\left(f(x)=\frac{6}{x^2+3}\right)$ has two points at which the concavity changes: $x=-1, x=1$. If the tangent line to the graph exists at such point, that point is a **point of inflection**.

$$Point\ of\ Inflection\begin{cases} f''(x)=0 \\ tangent\ line\ exist \end{cases}$$

Three types of points of inflection



Example3: Determine the points of inflection and discuss the concavity of the graph of $f(x)=x^4-4x^3$

Solution

Step 1: Domain

$$f(x)=x^4-4x^3\quad x\in R$$

Step 2: $f'(x)$

$$f'(x)=4x^3-12x^2\quad x\in R$$

Step 3: $f''(x)$

$$f''(x)=12x^2-24x=0\quad x\in R$$

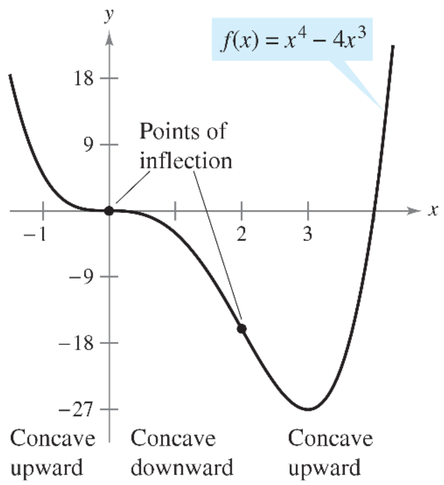
$$x=0, x=2$$ inflection point

Step 4: test value in each interval

$-\infty<x<0, f''(-1)>0$: upward

$0<x<2, f''(1)<0$: downward

$2<x<\infty, f''(3)>0$: upward



Points of inflection can occur where $f''(x)=0$ or f'' does not exist.

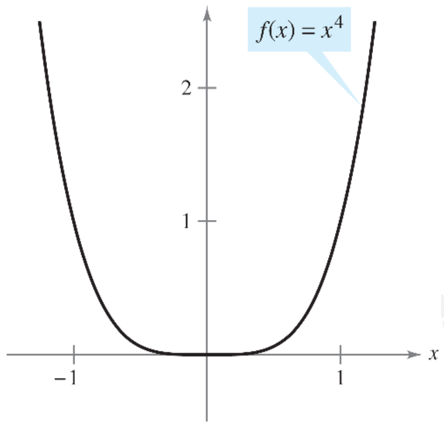
Important Notice: the graph of

$$f(x)=x^4$$

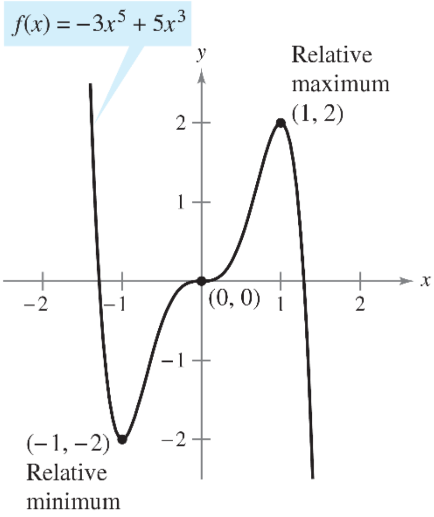
$$f''(x)=0$$

but $(0,0)$ is not a inflection point.

<u>四次方以上的幂函数需要注意。</u>



$f''(x)=0$, but $(0, 0)$ is not a point of inflection.



(0, 0) is neither a relative minimum nor a relative maximum.

Example 4: Find the relative extrema for $f(x) = -3x^5 + 5x^3$

Solution

Step 1: Domain

$$f(x) = -3x^5 + 5x^3 \quad x \in R$$

Step 2: $f'(x)$

$$f'(x) = -15x^4 + 15x^2$$
$$= 15x^2(1 - x^2)$$
$$x = -1, 0, 1$$

Step 3: $f''(x)$

$$f''(x) = -60x^3 + 30x$$
$$= 30x(-2x^2 + 1)$$
$$x = 0, x = \pm\frac{1}{\sqrt{2}}$$ inflection point

Step 4: test value in each interval

Point	$(-1, -2)$	$(1, 2)$	$(0, 0)$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(0) = 0$
Conclusion	Relative minimum	Relative maximum	Test fails

Notice: at $(0,0)$, $f''(0) = 0$, so test fails

Exercise: find the points of inflection and discuss the concavity of the graph of the function.

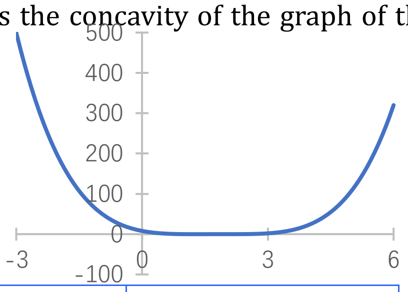
26. $f(x) = (x-2)^3(x-1)$

$f'(x) = (x-2)^2(3x-2)$

$f''(x) = (x-2)(9x-12) = 0$

$\Rightarrow \ x = 2, x = 4/3$

Notice: at x=2, $f'(x) = 0$ and $f''(x) = 0$



Interval	$-\infty < x < 4/3$	$4/3 < x < 2$	$2 < x < \infty$
Test value	$x = 0$	$x = 1$	$x = 3$
Sign of $f''(x)$	$f''(0) > 0$	$f''(1) > 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave upward	Concave upward

27. $f(x) = x\sqrt{x+3}$

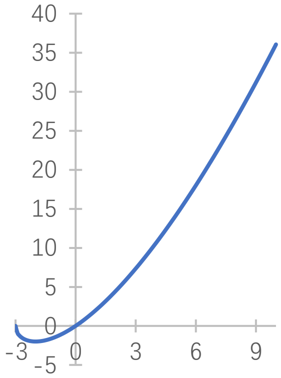
$f(x) = x\sqrt{x+3}$ $\ Domain: x \geq -3$ Step 1: Domain

$f'(x) = \frac{3x/2+3}{(x+3)^{1/2}}$ Step 2: $f'(x)$

$f''(x) = \frac{3x/4+3}{(x+3)^{3/2}} = 0 \quad \Rightarrow \text{x} = -4$ Step 3: $f''(x)$

$x = -4$ is out of domain

$-3 < x < \infty, f''(0) > 0$: concave upward Step 4: test value



28. $f(x) = x\sqrt{9-x}$

$f(x) = x\sqrt{9-x}$ $\ Domain: x \leq 9$ Step 1: Domain

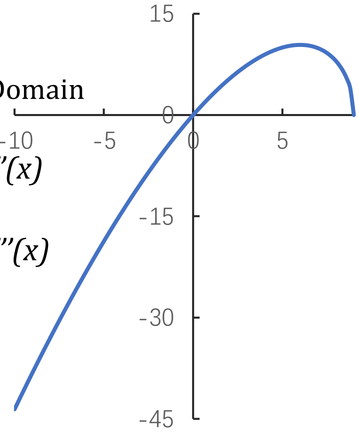
$f'(x) = \frac{9-3x/2}{\sqrt{9-x}}$ Step 2: $f'(x)$

$f''(x) = \frac{3x-36}{4(9-x)^{3/2}} = 0 \quad \Rightarrow \text{x} = 12$ Step 3: $f''(x)$

$x = 12$ is out of domain

Step 4: test value in each interval

$-\infty < x < 9, f''(0) < 0$: concave downward



29. $f(x) = \frac{4}{x^2+1}$

$f(x) = \frac{4}{x^2+1} \quad x\epsilon R$

$f'(x) = -4(x^2+1)^{-2}$

$f''(x) = 8(x^2+1)^{-3} = 0$ no solution

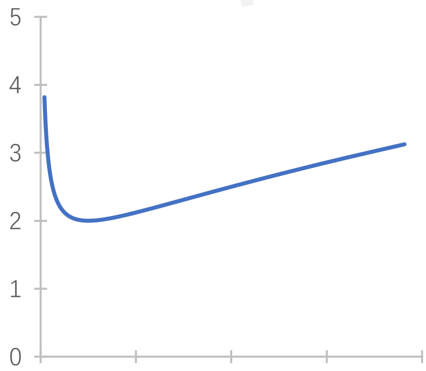
$f''(0) => 0$ concave downward

30. $f(x) = \frac{x+1}{\sqrt{x}}$

$f(x) = \frac{x+1}{\sqrt{x}} = x^{1/2} + x^{-1/2} \quad x > 0$

$f'(x) = \frac{1}{2}x^{\frac{-1}{2}} - \frac{1}{2}x^{\frac{-3}{2}}$

$f''(x) = \frac{-1}{4}x^{\frac{-3}{2}} + \frac{3}{4}x^{\frac{-5}{2}} = 0$



$x^{\frac{-3}{2}}(3x^{-1}-1)=0 \Rightarrow x=3$

$0<x<3, f''(1)>0$: concave upward
$3<x<\infty, f''(6)<0$: concave downward

Exercise: 比较下列方程式的大小

1. $\sqrt{x+1}-\sqrt{x}$ 与 $\sqrt{x}-\sqrt{x-1}$ $(x>2)$

2. $\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{x}}$ 与 $\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x-1}}$ $(x>1)$

高中方法：分子或分母有理化

微积分方法：1 题是$y=\sqrt{x}$，$\Delta x=1, f''<0$凸函数Δy越来越小，2 题是$y=\frac{1}{\sqrt{x}}, \Delta x=1, f''>0$凹函数$\Delta y$越来越大.

Exercise: find the points of inflection and discuss the concavity of the graph of the function.

26. $f(x) = (x-2)^3(x-1)$

Interval			
Test value			
Sign of $f''(x)$			
Conclusion			

27. $f(x) = x\sqrt{x+3}$

28. $f(x) = x\sqrt{9-x}$

29. $f(x) = \frac{4}{x^2+1}$

30. $f(x) = \frac{x+1}{\sqrt{x}}$

Exercise: 比较下列方程式的大小

1. $\sqrt{x+1}-\sqrt{x}$ 与 $\sqrt{x}-\sqrt{x-1}$ $(x>2)$

2. $\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{x}}$ 与 $\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x-1}}$ $(x>1)$